Taylor Series

Questions

Q1.

$$\left[\begin{array}{c}\textit{The Taylor series expansion of } f(x) \textit{ about } x = a \textit{ is given by} \\ f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \ldots + \frac{(x-a)^r}{r!}f^{(r)}(a) + \ldots\end{array}\right]$$

Given that

$y = (1 + \ln x)2 \qquad x > 0$

(a) show that $\frac{d^2y}{dx^2} = -\frac{2\ln x}{x^2}$

(4)

(b) Hence find $\frac{d^3y}{dx^3}$

(2)

(c) Determine the Taylor series expansion about $x = 1$ of

$$(1 + \ln x)^2$$

in ascending powers of $(x - 1)$, up to and including the term in $(x - 1)^3$
Give each coefficient in simplest form.

(3)

(d) Use this series expansion to evaluate

$$\lim_{x\to 1} \frac{2x - 1 - (1 + \ln x)^2}{(x-1)^3}$$

explaining your reasoning clearly.

(3)

(Total for question = 12 marks)

Q2.

$$\left[\begin{array}{c}\textit{The Taylor series expansion of } \mathrm{f}(x) \textit{ about } x = a \textit{ is given by} \\ \mathrm{f}(x) = \mathrm{f}(a) + (x-a)\mathrm{f}'(a) + \frac{(x-a)^2}{2!}\mathrm{f}''(a) + \ldots + \frac{(x-a)^r}{r!}\mathrm{f}^{(r)}(a) + \ldots\end{array}\right]$$

(i) (a) Use differentiation to determine the Taylor series expansion of ln *x*, in ascending powers of (x – 1) , up to and including the term in (x – 1)

(4)

(b) Hence prove that

$$\lim_{x\to 1}\left(\frac{\ln x}{x-1}\right) = 1$$

(2)

(ii) Use L'Hospital's rule to determine

$$\lim_{x\to 0}\left(\frac{1}{(x+3)\tan(6x)\operatorname{cosec}(2x)}\right)$$

(*Solutions relying entirely on calculator technology are not acceptable*.)

(4)

(Total for question = 10 marks)

Q3.

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + y = 0 \qquad \text{(I)}$$

(a) Show that

$$\frac{d^5y}{dx^5} = ax\frac{d^4y}{dx^4} + b\frac{d^3y}{dx^3}$$

where *a* and *b* are integers to be found.

(4)

(b) Hence find a series solution, in ascending powers of *x*, as far as the term in x^5,

of the differential equation (I) where y = 0 and $\frac{dy}{dx}$ = at x = 0

(5)

(Total for question = 9 marks)

Q4.

$$f(x) = x^4 \sin(2x)$$

Use Leibnitz's theorem to show that the coefficient of $(x - \pi)^8$ in the Taylor series expansion of f (x) about π is

$$\frac{a\pi + b\pi^3}{315}$$

where a and b are integers to be determined.

(8)

$$\left[\begin{array}{c}\textit{The Taylor series expansion of } f(x) \textit{ about } x = k \textit{ is given by} \\ f(x) = f(k) + (x-k)f'(k) + \frac{(x-k)^2}{2!}f''(k) + \ldots + \frac{(x-k)^r}{r!}f^{(r)}(k) + \ldots\end{array}\right]$$

(Total for question = 8 marks)

Q5.

$$\frac{dy}{dx} = x - y^2 \qquad \text{(I)}$$

(a) Show that

$$\frac{d^5y}{dx^5} = ay\frac{d^4y}{dx^4} + b\frac{dy}{dx}\frac{d^3y}{dx^3} + c\left(\frac{d^2y}{dx^2}\right)^2$$

where a, b and c are integers to be determined.

(4)

(b) Hence find a series solution, in ascending powers of x as far as the term in x^5, of the differential equation (I), given that $y = 1$ at $x = 0$

(5)

(Total for question = 9 marks)

Mark Scheme – Taylor Series

Q1.

Question	Scheme	Marks	AOs
(a)	$y=(1+\ln x)^2 \Rightarrow \frac{dy}{dx}=k(1+\ln x)\times\frac{1}{x}$ or $y=1+2\ln x+(\ln x)^2 \Rightarrow \frac{dy}{dx}=\frac{A}{x}+B\ln x\times\frac{1}{x}$	M1	1.1b
	$y=(1+\ln x)^2 \Rightarrow \frac{dy}{dx}=2(1+\ln x)\times\frac{1}{x}$ or $y=1+2\ln x+(\ln x)^2 \Rightarrow \frac{dy}{dx}=\frac{2}{x}+2\ln x\times\frac{1}{x}$	A1	1.1b
	$\frac{d^2y}{dx^2}=\frac{k\left(\frac{1}{x}\right)\times x-k(1+\ln x)\times 1}{x^2}$ or $\frac{k}{x}x^{-1}+k(1+\ln x)(-x^{-2})$ or $\frac{d^2y}{dx^2}=-\frac{A}{x^2}+\frac{\frac{B}{x}\times x-2\ln x\times 1}{x^2}$	M1	1.1b
	$\frac{d^2y}{dx^2}=\frac{\left(\frac{2}{x}\right)\times x-2(1+\ln x)}{x^2}$ or $\frac{2}{x}x^{-1}+2(1+\ln x)(-x^{-2})$ Leading to $\frac{d^2y}{dx^2}=-\frac{2\ln x}{x^2}$* achieved from correct work.	A1*	2.1
		(4)	
(b)	$\frac{d^3y}{dx^3}=\frac{\pm\frac{C}{x}\times x^2\pm Dx\ln x}{x^4}$ or $\frac{d^3y}{dx^3}=(-2\ln x)(-Cx^{-3})+\left(-\frac{D}{x}\right)(x^{-2})$	M1	1.1b
	$\frac{d^3y}{dx^3}=-\frac{\frac{2}{x}\times x^2-4x\ln x}{x^4}$ or $\frac{-\frac{2}{x}\times x^2-(-2\ln x)(2x)}{x^4}$ or $-\frac{2}{x^3}+\frac{4\ln x}{x^3}$	A1	1.1b
		(2)	
(c)	$y(1)=1, y'(1)=2, y''(1)=0, y'''(1)=-2$	M1	1.1b
	$[y]=1+2(x-1)+\frac{0}{2!}(x-1)^2+\frac{-2}{3!}(x-1)^3+...$	M1	2.5
	$[y]=1+2(x-1)-\frac{1}{3}(x-1)^3+...$ or $[y]=-1+2x-\frac{1}{3}(x-1)^3+...$	A1	1.1b
		(3)	
(d)	$\frac{2x-1-(1+\ln x)^2}{(x-1)^3}=\frac{2x-1-1-2(x-1)+\frac{1}{3}(x-1)^3+...}{(x-1)^3}=\frac{\frac{1}{3}(x-1)^3+...}{(x-1)^3}$	M1	1.1b

	Simplifies and realises that terms cancel to leave a constant term $\dfrac{\frac{1}{3}(x-1)^3+\ldots}{(x-1)^3}=\frac{1}{3}$	M1	3.1a
	Hence $\lim_{x\to 1}\dfrac{2x-1-(1+\ln x)^2}{(x-1)^3}=\dfrac{1}{3}$ as all remaining terms will become zero in the limit as they are multiples of $(x-1)^k$, which tends to 0.	A1	2.4
		(3)	
			(12 marks)

Notes:

(a)

M1: Attempts the first derivative, including use of the chain rule. May expand first. E.g. accept forms as shown.

A1: Correct first derivative, need not be simplified.

M1: Attempts second derivative using quotient rule or product rule – examples as shown, or equivalents accepted.

A1*: Correct result achieved from correct work.

(b)

M1: Applies quotient rule or product rule to achieve third derivative. If formula is quoted it must be correct, if not accept derivatives of the form shown as there may be confusion with the minus sign.

A1: Correct third derivative, any form.

(c)

M1: Find value of derivatives at $x = 1$.

M1: Applies Taylor series expansion

A1: Correct series, may be unsimplified, isw once correct series seen. Must be using a correct third derivative.

(d)

M1: Applies the series to the limit and cancels terms in numerator to leave term in $(x-1)^3$ and above only (may not see +… for this mark)

M1: Simplifies and realises that the $(x-1)^3$ cancels and achieves a constant A

A1: Correct limit deduced with reasoning given why the remaining terms disappear.

Q2.

Question	Scheme	Marks	AOs
(i) (a)	$f(x) = ln\,x \quad \Rightarrow f(1) = 0$ $f'(x) = \frac{1}{x} \quad \Rightarrow f'(1) = 1$ $f^{''}\frac{1}{x^2}f^{''}$	M1 A1	1.1b 1.1b
	$(ln\,x) = (0+)(x-1) - \frac{1}{2}(x-1)^2 + \ldots$	M1 A1	2.5 1.1b
		(4)	
(i) (b)	$\lim_{x \to 1}\left(\frac{ln\,x}{x-1}\right) = \lim_{x \to 1}\left(\frac{(x-1) - \frac{1}{2}(x-1)^2 + \ldots}{x-1}\right)$ $= \lim_{x \to 1}\left(1 - \frac{1}{2}(x-1) + \ldots\right)$	M1	2.1
	$= \lim_{x \to 1}\left(1 - \frac{1}{2}(x-1) + \ldots\right) = 1^*$ cso	A1*	2.2a
		(2)	
(ii)	Writes as an indeterminate form For example $\frac{sin(2x)}{(x+3)\,tan(6x)}$ or $\frac{sin(2x)\,cos(6x)}{(x+3)\,sin(6x)}$	M1	3.1a
	Differentiates numerator and denominator using appropriate rules $\frac{2\,cos(2x)}{tan(6x)+6(x+3)\,sec^2(6x)}$ or $\frac{2\,cos(2x)\,cos(6x)-6\,sin(2x)\,sin(6x)}{sin(6x)+6(x+3)\,cos(6x)}$	M1 A1	1.1b 1.1b
	$\lim_{x \to 0}\left(\frac{1}{(x+3)\,tan(6x)\text{cosec}(2x)}\right) = \frac{2}{18} = \frac{1}{9}$ o.e	A1cso	2.2a
		(4)	
			(10 marks)

Notes:
(i) (a) Notes: ignore extra terms throughout.   **M1:** Differentiates $f(x) = ln\,x$ twice and finds $f(1), f'(1)$ and $f^{''(1)}$ **A1:** Correct differentiation and values for $f(1), f'(1)$ and $f^{''(1)}$.   **M1:** Uses correct mathematical notation to find the Taylor series for $ln\,x$ in powers of $(x-1)$ up to $(x-1)^2$ **A1:** Correct expansion with simplified coefficients. Do not be concerned with the left hand side.
(i) (b) **Question says "hence" so the result of (a) must be used.** No marks for l'Hosptial's rule on the original functions (send to review if attempted with their part (a)).   **M1:** Substitutes their Taylor series for $ln\,x$ in powers of $(x-1)$ up to $(x-1)^2$ into the limit and cancels a factor $(x-1)$ from each term. Allow for the cancelling seeing a relevant strikethrough in all $x-1$ terms.   **A1*:** $\lim_{x \to 1}\left(\frac{ln\,x}{x-1}\right) = \lim_{x \to 1}\left(1 - \frac{1}{2}(x-1) + \ldots\right) = 1$ cso Must have come from a correct expansion.   Must see the $1 - \frac{1}{2}(x-1)$
(ii)

M1: Writes the fraction in an indeterminate form $\frac{f(x)}{g(x)}$ where $\frac{f(0)}{g(0)} = \frac{0}{0}$ or $\frac{f(0)}{g(0)} = \frac{"\infty"}{"\infty"}$

M1: Differentiates numerator and denominator using appropriate rules, ie product rule for a product etc. Allow slips in coefficients but the form should be correct. This mark is available as long as written as $\frac{f(x)}{g(x)}$ even if not an indeterminate form. May be seen written as separate from the fraction, ie f'(x) = … and g'(x) = …

A1: Depend on both Ms. Correct differentiation for derivatives that lead to a limit. Must have a derivative for which g'(0) ≠ 0 and is finite.

A1cso: Deduces the correct limit from fully correct work.

Q3.

Question	Scheme	Marks	AOs
(a)	$y'' = 2xy' - y \Rightarrow y''' = 2xy'' + 2y' - y'$	M1	1.1b
		A1	1.1b
	$y''' = 2xy'' + y' \Rightarrow y'''' = 2xy''' + 2y'' + y''$	M1	2.1
	$y'''' = 2xy''' + 3y'' \Rightarrow y''''' = 2xy'''' + 5y'''$	A1	2.1
		(4)	
(b)	$x = 0, y = 0, y' = 1 \Rightarrow y''(0) = 0$ from equation	B1	2.2a
	$y'''(0) = 2\times 0\times y''(0) + 1 = 1;\quad y''''(0) = 2\times 0\times 1 + 3\times 0 = 0;$	M1	1.1b
	$x = 0,\ y'''(0) = 1,\ y''''(0) = 0 \Rightarrow y'''''(0) = 5$	A1	1.1b
	$y = y(0) + y'(0)x + \frac{y''(0)}{2}x^2 + \frac{y'''(0)}{6}x^3 + \frac{y''''(0)}{24}x^4 + \frac{y'''''(0)}{120}x^5 + \ldots$	M1	2.5
	Series solution: $y = x + \frac{1}{6}x^3 + \frac{1}{24}x^5 + \ldots$	A1ft	1.1b
		(5)	
		(9 marks)	

Notes

(a)

M1 Attempts to differentiate equation with use of the product rule.

A1 cao. Accept if terms all on one side.

M1 Continues the process of differentiating to progress towards the goal. Terms may be kept on one side, but an expression in the fourth derivative should be obtained.

A1 Completes the process to reach the fifth derivative and rearranges to the correct form to obtain the correct answer by correct solution only.

(b)

B1 Deduces the correct value for $y''(0)$ from the information in the question.

M1 Finds the values of the derivatives at the given point.

A1 All correct

M1 Correct mathematical language required with given denominators. Can be in factorial form.

A1ft Correct series, must start $y = \ldots$. Follow through the values of their derivatives at 0.

Q4.

Question	Scheme	Marks	AOs
	$f(x) = x^4 \sin(2x) \quad u = x^4 \quad v = \sin(2x)$		
	$u' = 4x^3, u'' = 12x^2, u''' = 24x, u^{(4)} = 24$ (and $u^{(n)} = 0$ for $n > 4$)	M1	1.1b
	$v' = 2\cos(2x), v'' = -4\sin(2x), v''' = -8\cos(2x), v^{(4)} = 16\sin(2x),$ $v^{(5)} = 32\cos(2x), v^{(6)} = -64\sin(2x), v^{(7)} = -128\cos(2x),$ $v^{(8)} = 256\sin(2x),$	M1 A1 A1	3.1a 1.1b 1.1b
	Thus $f^{(8)}(x) = x^4 \times 256\sin(2x) + 8 \times 4x^3 \times -128\cos(2x) + \frac{8\times 7}{2} \times 12x^2 \times -64\sin(2x) + \frac{8\times 7\times 6}{6} \times 24x \times 32\cos(2x) + \frac{8\times 7\times 6\times 5}{24} \times 24 \times 16\sin(2x)$ $f^{(8)}(x) = x^4 \times 256\sin(2x) + 8 \times 4x^3 \times -128\cos(2x)$ $+28 \times 12x^2 \times -64\sin(2x) + 56 \times 24x \times 32\cos(2x)$ $+70 \times 24 \times 16\sin(2x)$	M1	2.1
	$f^{(8)}(\pi) = 0 - 4096\pi^3 - 0 + 1344 \times 2^5 \pi + 0 \; (= -4096\pi^3 + 43008\pi)$	M1	1.1b
	Coefficient is $\frac{f^{(8)}(\pi)}{8!} = \frac{1344 \times 2^5 \pi - 4096\pi^3}{8! \text{ or } \{40320\}}$	M1	2.2a
	$= \frac{336\pi - 32\pi^3}{315}$ (So $a = 336$ and $b = -32$)	A1	2.1
		(8)	
			(8 marks)

Notes:
M1: Establishes the non-disappearing derivatives of x^4. Allow slips in coefficients, but powers must decrease. **M1:** Identifies the relevant derivatives for $\sin(2x)$, up to the 8[th] derivative or establishes the correct pattern. Look for alternating between sin and cos. Condone use of x. **A1:** Correct sizes for the coefficients, allow sign errors for this mark (may be due to incorrect signs when differentiating sin and cos) Must have angle $2x$. **A1:** All derivatives correctly established. (Note the sin terms may be omitted if the student has made clear they will disappear, but if present they must be correct). **M1:** Applies Leibnitz's theorem to get the 8[th] derivative with their expressions. Binomial coefficients must be present. **M1:** Evaluates their 8[th] derivative at π **M1:** Uses Taylor series – divides their value for $f^{(8)}(\pi)$ by 8! **A1:** Simplifies to the correct answer. **Note: If do not use Leibnitz's theorem then maximum M0 M0 A0 A0 M0 M1 M1 A0**

Q5.

Question	Scheme	Marks	AOs
(a)	$\frac{d^2y}{dx^2}=1-2y\frac{dy}{dx}\Rightarrow\frac{d^3y}{dx^3}=-2y\frac{d^2y}{dx^2}-2\left(\frac{dy}{dx}\right)^2$	M1 A1	1.1b 1.1b
	$\frac{d^4y}{dx^4}=-2\frac{dy}{dx}\frac{d^2y}{dx^2}-2y\frac{d^3y}{dx^3}-4\frac{dy}{dx}\frac{d^2y}{dx^2}=-6\frac{dy}{dx}\frac{d^2y}{dx^2}-2y\frac{d^3y}{dx^3}$	**d**M1	2.1
	$\frac{d^5y}{dx^5}=-6\frac{dy}{dx}\frac{d^3y}{dx^3}-6\left(\frac{d^2y}{dx^2}\right)^2-2y\frac{d^4y}{dx^4}-2\frac{dy}{dx}\frac{d^3y}{dx^3}$ $=-2y\frac{d^4y}{dx^4}-8\frac{dy}{dx}\frac{d^3y}{dx^3}-6\left(\frac{d^2y}{dx^2}\right)^2$	A1	2.1
		(4)	
(b)	$x=0, y=1\Rightarrow\left(\frac{dy}{dx}\right)_0=0-1^2=-1$	B1	2.2a
	$\left(\frac{d^2y}{dx^2}\right)_0=1-2(1)(-1)=3, \left(\frac{d^3y}{dx^3}\right)_0=-2(1)(3)-2(-1)^2=-8$ $\left(\frac{d^4y}{dx^4}\right)_0=-6(-1)(3)-2(1)(-8)=34,$ $\left(\frac{d^5y}{dx^5}\right)_0=-2(1)(34)-8(-1)(-8)-6(3)^2=-186$	M1 A1	1.1b 1.1b
	$y=y(0)+x\left(\frac{dy}{dx}\right)_0x+\frac{x^2}{2!}\left(\frac{d^2y}{dx^2}\right)_0+\frac{x^3}{3!}\left(\frac{d^3y}{dx^3}\right)_0+\frac{x^4}{4!}\left(\frac{d^4y}{dx^4}\right)_0+\frac{x^5}{5!}\left(\frac{d^5y}{dx^5}\right)_0+\ldots$ With their values	M1	2.5
	$(y=)1-x+\frac{3}{2}x^2-\frac{8}{6}x^3+\frac{34}{24}x^4-\frac{186}{120}x^5+\ldots$ $(y=)1-x+\frac{3}{2}x^2-\frac{4}{3}x^3+\frac{17}{12}x^4-\frac{31}{20}x^5+\ldots$	A1ft	1.1b
		(5)	
		(9 marks)	

(a)

M1: Attempts to find the second and third derivatives:

This requires $\frac{d^2y}{dx^2}=1\pm2y\frac{dy}{dx}$ or $\frac{d^2y}{dx^2}=\pm2y\frac{dy}{dx}$ followed by $\frac{d^3y}{dx^3}=\pm2y\frac{d^2y}{dx^2}\pm\ldots$ or $\frac{d^3y}{dx^3}=\pm\ldots\pm2\left(\frac{dy}{dx}\right)^2$

A1: Correct second **and** third derivatives.

dM1: Continues to differentiate to reach the 5th derivative. **This is dependent on the first method mark but there is no need to check the detail and the mark can be awarded as long as the 5th derivative is reached.**

A1: Completes the process, collecting terms if necessary, to obtain the correct expression (NB $a=-2$, $b=-8$, $c=-6$)

Allow dash/dot notation for the derivatives but the final answer must be in the correct form.

Note that if $\frac{d^2y}{dx^2} = \pm 2y\frac{dy}{dx}$ is obtained initially, allow a full recovery in (a).

Note that (a) can be found using Leibnitz's theorem and the following scheme should be applied:

M1: $\frac{d^2y}{dx^2} = 1 \pm 2y\frac{dy}{dx}$ or $\frac{d^2y}{dx^2} = \pm 2y\frac{dy}{dx}$ followed by an attempt to differentiate y 3 times and $\frac{dy}{dx}$ 3 times.

A1: All correct

dM1: $\frac{d^5y}{dx^5} = -2\frac{dy}{dx}\frac{d^4y}{dx^4} - 3\times 2\frac{dy}{dx}\frac{d^3y}{dx^3} - 3\times 2\left(\frac{d^2y}{dx^2}\right)^2 - 2y\frac{d^3y}{dx^3}\frac{dy}{dx}$ (correct application of Leibnitz)

A1: $= -2y\frac{d^4y}{dx^4} - 8\frac{dy}{dx}\frac{d^3y}{dx^3} - 6\left(\frac{d^2y}{dx^2}\right)^2$

As in the main scheme, if $\frac{d^2y}{dx^2} = \pm 2y\frac{dy}{dx}$ is obtained initially, allow a full recovery in (a).

Alternative for (a):

M1: $\frac{d^2y}{dx^2} = 1 - 2y\frac{dy}{dx} = 1 - 2y\left(x - y^2\right) = 1 - 2xy + 2y^3 \Rightarrow \frac{d^3y}{dx^3} = -2y - 2x\frac{dy}{dx} + 6y^2\frac{dy}{dx}$

Score for the second derivative form as in the main scheme and then an attempt at the third derivative with at least 2 terms correct.

A1: Fully correct

Then as main scheme.

(b)

B1: Deduces the correct value for $y'(0)$

M1: Finds the values of all the other derivatives at $x = 0$ up to 5th. There is no need to check their values as long as there is no obvious incorrect work, but values for all the derivatives up to the 5th must be found

A1: All values correct (as single values – e.g. do not allow unsimplified)

M1: Applies the correct Maclaurin series for their values including the factorials up to the term in x^5

A1ft: Correct expansion, follow through their values for the derivatives. This does not have to be simplified but the factorials need to be evaluated. **Once a correct, or correct follow through expression is seen apply isw.**